

PIATETSKI-SHAPIRO PRIMES IN A BEATTY SEQUENCE

VICTOR Z. GUO
Department of Mathematics
University of Missouri
Columbia, MO 65211 USA
zgbmf@math.missouri.edu

Abstract

Let α, β be real numbers such that $\alpha > 1$ is irrational and of finite type, and let c be a real number in the range $1 < c < \frac{14}{13}$. In this paper, it is shown that there are infinitely many Piatetski-Shapiro primes $p = \lfloor n^c \rfloor$ in the non-homogenous Beatty sequence $(\lfloor \alpha m + \beta \rfloor)_{m=1}^{\infty}$.

1 Introduction

For fixed real numbers α, β the associated *non-homogeneous Beatty sequence* is the sequence of integers defined by

$$\mathcal{B}_{\alpha, \beta} = (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called *generalized arithmetic progressions*. It is known that there are infinitely many prime numbers in the Beatty sequence if $\alpha > 0$ (see, for example, the proof of Ribenboim [7, p. 289]). Moreover, if $\alpha \geq 1$, then the counting function

$$\pi_{\alpha, \beta}(x) = \#\{\text{prime } p \leq x : p \in \mathcal{B}_{\alpha, \beta}\}$$

satisfies the asymptotic relation

$$\pi_{\alpha, \beta}(x) \sim \frac{x}{\alpha \log x} \quad \text{as } x \rightarrow \infty.$$

The *Piatetski-Shapiro sequences* are sequences of the form

$$\mathcal{N}^{(c)} = (\lfloor n^c \rfloor)_{n=1}^{\infty} \quad (c > 1, c \notin \mathbb{N}).$$

Such sequences have been named in honor of Piatetski-Shapiro, who proved [6] that $\mathcal{N}^{(c)}$ contains infinitely many primes if $c \in (1, \frac{12}{11})$. More precisely, for such c he showed that the counting function

$$\pi^{(c)}(x) = \#\{\text{prime } p \leq x : p \in \mathcal{N}^{(c)}\}$$

satisfies the asymptotic relation

$$\pi^{(c)}(x) \sim \frac{x^{1/c}}{c \log x} \quad \text{as } x \rightarrow \infty.$$

The admissible range for c in this asymptotic formula has been extended many times over the years and is currently known to hold for all $c \in (1, \frac{243}{205})$ thanks to Rivat and Wu [8]. The same result is expected to hold for all larger values of c . We remark that if $c \in (0, 1)$ then $\mathcal{N}^{(c)}$ contains all natural numbers, hence all primes in particular.

Since both sequences $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{N}^{(c)}$ contain infinitely many primes in the cases described above, it is natural to ask whether infinitely many primes lie in the intersection $\mathcal{B}_{\alpha,\beta} \cap \mathcal{N}^{(c)}$ in some instances. In this paper we answer this question in the affirmative for certain values of the parameters α, β, c . Our main result is the following quantitative theorem.

Theorem 1. *Let $\alpha, \beta \in \mathbb{R}$, and suppose that $\alpha > 1$ is irrational and of finite type. Let $c \in (1, \frac{14}{13})$. There are infinitely many primes in both the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ and the Piatetski-Shapiro sequence $\mathcal{N}^{(c)}$. Moreover, the counting function*

$$\pi_{\alpha,\beta}^{(c)}(x) = \#\{\text{prime } p \leq x : p \in \mathcal{B}_{\alpha,\beta} \cap \mathcal{N}^{(c)}\}$$

satisfies

$$\pi_{\alpha,\beta}^{(c)}(x) = \frac{x^{1/c}}{\alpha c \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right),$$

where the implied constant depends only on α and c .

Remarks. We recall that the type $\tau = \tau(\alpha)$ of the irrational number α is defined by

$$\tau = \sup \left\{ t \in \mathbb{R} : \liminf_{n \rightarrow \infty} n^t \llbracket \alpha n \rrbracket = 0 \right\},$$

where $\llbracket t \rrbracket$ denotes the distance from a real number t to the nearest integer. For technical reasons we assume that α is of finite type in the statement of the theorem; however, we expect the result holds without this restriction.

If α is a rational number, then the Beatty sequence $\mathcal{B}_{\alpha,\beta}$ is a finite union of arithmetic progressions. In the case, Theorem 1 also holds (in a wider range of c) thanks to the work of Leitmann and Wolke [11], who showed that for any coprime integers a, d with $1 \leq a \leq d$ and any real number $c \in (1, \frac{12}{11})$ the counting function

$$\pi^{(c)}(x; d, a) = \#\{p \leq x : p \in \mathcal{N}^{(c)} \text{ and } p \equiv a \pmod{d}\},$$

satisfies

$$\pi_c(x; d, a) \sim \frac{x^{1/c}}{\phi(d) \log(x)} \quad \text{as } x \rightarrow \infty, \tag{1}$$

where ϕ is the Euler function (a more explicit relation than (1) holds in the shorter range $1 < c < \frac{18}{17}$; see Baker *et al* [1, Theorem 8]).

We also remark that our theorem is only stated for real numbers $\alpha > 1$, for if $\alpha \in (0, 1]$ then the set $\mathcal{B}_{\alpha, \beta}$ contains all but finitely many natural numbers.

2 Preliminaries

2.1 Notation

We denote by $[t]$ and $\{t\}$ the integer part and the fractional part of t , respectively. As is customary, we put

$$\mathbf{e}(t) = e^{2\pi it} \quad \text{and} \quad \{t\} = t - [t] \quad (t \in \mathbb{R}).$$

Throughout the paper, we make considerable use of the sawtooth function defined by

$$\psi(t) = t - [t] - \frac{1}{2} = \{t\} - \frac{1}{2} \quad (t \in \mathbb{R})$$

For the Beatty sequence $\mathcal{B}_{\alpha, \beta} = ([\alpha n + \beta])_{n=1}^{\infty}$ we systematically denote $a = \alpha^{-1}$ and $b = \alpha^{-1}(1 - \beta)$. For the Piatetski-Shapiro sequence $([n^c])_{n=1}^{\infty}$ we always put $\gamma = 1/c$.

Throughout, the letter p always denotes a prime.

Implied constants in the symbols O and \ll may depend on the parameters c and A (where obvious) but are absolute otherwise. We use notation of the form $m \sim M$ as an abbreviation for $M < m \leq 2M$.

For any set E of real numbers, we denote by \mathcal{X}_E the characteristic function of E ; that is,

$$\mathcal{X}_E(n) = \begin{cases} 1 & \text{if } n \in E, \\ 0 & \text{if } n \notin E. \end{cases}$$

2.2 Discrepancy

The *discrepancy* $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, a_2, \dots, a_M \in [0, 1)$ is defined by

$$D(M) = \sup_{\mathcal{I} \subseteq [0,1)} \left| \frac{V(\mathcal{I}, M)}{M} - |\mathcal{I}| \right|, \quad (2)$$

where the supremum is taken over all intervals \mathcal{I} contained in $[0, 1)$, $V(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}|$ is the length of the interval \mathcal{I} .

For any irrational number θ the sequence of fractional parts $(\{n\theta\})_{n=1}^\infty$ is uniformly distributed over $[0, 1)$ (see, e.g., [5, Example 2.1, Chapter 1]). In the special case that θ is of finite type, the following more precise statement holds (see [5, Theorem 3.2, Chapter 2]).

Lemma 1. *Let θ be a fixed irrational number of finite type τ . Then, for every $\theta \in \mathbb{R}$ the discrepancy $D_{\theta, \mu}(M)$ of the sequence $(\{\theta m + \mu\})_{m=1}^M$ satisfies the bound*

$$D_{\theta, \mu}(M) \leq M^{-1/\tau + o(1)} \quad (M \rightarrow \infty),$$

where the function implied by $o(\cdot)$ depends only on θ .

2.3 Lemmas

The following lemma provides a convenient characterization of the numbers that occur in the Beatty sequence $\mathcal{B}_{\alpha, \beta}$.

Lemma 2. *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha > 1$. Then*

$$n \in \mathcal{B}_{\alpha, \beta} \quad \Longleftrightarrow \quad \mathcal{X}_a(an + b) = 1$$

where \mathcal{X}_a is the periodic function defined by

$$\mathcal{X}_a(t) = \mathcal{X}_{(0, a]}(\{t\}) = \begin{cases} 1 & \text{if } 0 < \{t\} \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

By a classical result of Vinogradov (see [10, Chapter I, Lemma 12]) we have the following approximation of \mathcal{X}_a by a Fourier series.

Lemma 3. *For any $\Delta \in (0, \frac{1}{8})$ with $\Delta \leq \frac{1}{2} \min\{a, 1-a\}$, there is a real-valued function Ψ with the following properties:*

- (i) Ψ is periodic with period one;
- (ii) $0 \leq \Psi(t) \leq 1$ for all $t \in \mathbb{R}$;
- (iii) $\Psi(t) = \mathcal{X}_a(t)$ if $\Delta \leq \{t\} \leq a - \Delta$ or if $a + \Delta \leq \{t\} \leq 1 - \Delta$;
- (iv) $\Psi(t) = \sum_{k \in \mathbb{Z}} g(k) e(kt)$ for all $t \in \mathbb{R}$, where $g(0) = a$, and the other Fourier coefficients satisfy the uniform bound

$$g(k) \ll \min \{ |k|^{-1}, |k|^{-2} \Delta^{-1} \} \quad (k \neq 0). \quad (3)$$

We need the following well known approximation of Vaaler [9].

Lemma 4. *For any $H \geq 1$ there are numbers a_h, b_h such that*

$$\left| \psi(t) - \sum_{0 < |h| \leq H} a_h \mathbf{e}(th) \right| \leq \sum_{|h| \leq H} b_h \mathbf{e}(th), \quad a_h \ll \frac{1}{|h|}, \quad b_h \ll \frac{1}{H}.$$

Next, we recall the following identity for the von Mangoldt function Λ , which is due to Vaughan (see Davenport [3, p. 139]).

Lemma 5. *Let $U, V \geq 1$ be real parameters. For any $n > U$ we have*

$$\Lambda(n) = - \sum_{k|n} a(k) + \sum_{\substack{cd=n \\ d \leq V}} (\log c) \mu(d) - \sum_{\substack{kc=n \\ k > 1 \\ c > U}} \Lambda(c) b(k),$$

where

$$a(k) = \sum_{\substack{cd=k \\ c \leq U \\ d \leq V}} \Lambda(c) \mu(d) \quad \text{and} \quad b(k) = \sum_{\substack{d|k \\ d \leq V}} \mu(d)$$

We also need the following standard result; see [4, p. 48].

Lemma 6. *For a bounded function g and $N' \sim N$ we have*

$$\sum_{N < p \leq N'} g(p) \ll \frac{1}{\log N} \max_{N_1 \leq 2N} \left| \sum_{N < n \leq N_1} \Lambda(n) g(n) \right| + N^{1/2}.$$

We use the following result of Banks and Shparlinski [2, Theorem 4.1].

Lemma 7. *Let θ be a fixed irrational number of finite type $\tau < \infty$. Then, for every real number $0 < \varepsilon < 1/(8\tau)$, there is a number $\eta > 0$ such that the bound*

$$\left| \sum_{m \leq M} \Lambda(qm + a) \mathbf{e}(\theta km) \right| \leq M^{1-\eta}$$

holds for all integers $1 \leq k \leq M^\varepsilon$ and $0 \leq a < q \leq M^{\varepsilon/4}$ with $\gcd(a, q) = 1$ provided that M is sufficiently large.

We need the following lemma by Van der Corput; see [4, Theorem 2.2].

Lemma 8. *Let f be three times continuously differentiable on a subinterval \mathcal{I} of $(N, 2N]$. Suppose that for some $\lambda > 0$, the inequalities*

$$\lambda \ll |f''(t)| \ll \lambda \quad (t \in \mathcal{I})$$

hold, where the implied constants are independent of f and λ . Then

$$\sum_{n \in \mathcal{I}} \mathbf{e}(f(n)) \ll N\lambda^{1/2} + \lambda^{-1/2}.$$

We also need the following two lemmas for the bounds of certain type I and II sums. The two lemmas can be derived by revising the last three lines from the proofs of Baker *et al* [1, Lemma 24] and [1, Lemma 25], optimizing the ranges of K and L . Specifically we replace $1/3$ and $2/3$ into $3/7$ and $4/7$, respectively.

Lemma 9. *Suppose $|a_k| \leq 1$ for all $k \sim K$. Fix $\gamma \in (0, 1)$ and $m, h, d \in \mathbb{N}$. Then for any $K \ll N^{3/7}$ the type I sum*

$$S_I = \sum_{\substack{k \sim K \\ N < kl \leq N_1}} \sum_{l \sim L} a_k \mathbf{e}(mk^\gamma l^\gamma + khl/d)$$

satisfies the bound

$$S_I \ll m^{1/2} N^{3/7+\gamma/2} + m^{-1/2} N^{1-\gamma/2}.$$

Lemma 10. *Suppose $|a_k| \leq 1$ and $|b_l| \leq 1$ for $(k, l) \sim (K, L)$. Fix $\gamma \in (0, 1)$ and $m, h, d \in \mathbb{N}$. For any K in the range $N^{3/7} \ll K \ll N^{1/2}$, the type II sum*

$$S_{II} = \sum_{\substack{k \sim K \\ N < kl \leq N_1}} \sum_{l \sim L} a_k b_l \mathbf{e}(mk^\gamma l^\gamma + klh/d)$$

satisfies the bound

$$S_{II} \ll m^{-1/4} N^{1-\gamma/4} + m^{1/6} N^{16/21+\gamma/6} + N^{11/14}.$$

Finally, we use the following lemma, which provides a characterization of the numbers that occur in the Piatetski-Shapiro sequence $\mathcal{N}^{(c)}$.

Lemma 11. *A natural number m has the form $\lfloor n^c \rfloor$ if and only if $\mathcal{X}^{(c)}(m) = 1$, where $\mathcal{X}^{(c)}(m) = \lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor$. Moreover,*

$$\mathcal{X}^{(c)}(m) = \gamma m^{\gamma-1} + \psi(-m^\gamma) - \psi(-(m+1)^\gamma) + O(m^{\gamma-2}).$$

In particular, for any $c \in (1, \frac{243}{205})$ the results of [8] yield the estimate

$$\pi^{(c)}(x) = \sum_{p \leq x} \mathcal{X}^{(c)}(p) = \frac{x^\gamma}{c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right). \quad (4)$$

3 Construction

In what follows, we use τ to denote the (finite) type of α .

To begin, we express $\pi_{\alpha, \beta}^{(c)}(x)$ as a sum with the characteristic functions of the Beatty and Piatetski-Shapiro sequences; using Lemmas 2 and 11 we have

$$\pi_{\alpha, \beta}^{(c)}(x) = \sum_{p \leq x} \mathcal{X}_a(ap + b) \mathcal{X}^{(c)}(p).$$

In view of the properties (i)–(iii) of Lemma 3 it follows that

$$\pi_{\alpha,\beta}^{(c)}(x) = \sum_{p \leq x} \Psi(ap + b) \mathcal{X}^{(c)}(p) + O(V(\mathcal{I}, x)) \quad (5)$$

holds with some small $\Delta > 0$, where $V(\mathcal{I}, x)$ is the number of primes $p \in \mathcal{N}^{(c)}$ not exceeding x for which

$$\{ap + b\} \in \mathcal{I} = [0, \Delta) \cup (\alpha - \Delta, \alpha + \Delta) \cup (1 - \Delta, 1);$$

that is,

$$V(\mathcal{I}, x) = \sum_{p \leq x} \mathcal{X}_{\mathcal{I}}(\{ap + b\}) \mathcal{X}^{(c)}(p).$$

By Lemma 11 we see that

$$V(\mathcal{I}, x) = \gamma V_1(x) + V_2(x) + O(1),$$

where

$$\begin{aligned} V_1(x) &= \sum_{p \leq x} \mathcal{X}_{\mathcal{I}}(\{ap + b\}) p^{\gamma-1}, \\ V_2(x) &= \sum_{p \leq x} \mathcal{X}_{\mathcal{I}}(\{ap + b\}) (\psi(-p^\gamma) - \psi(-(p+1)^\gamma)). \end{aligned}$$

Using (4) we immediately derive the bound

$$V_2(x) \leq \sum_{p \leq x} (\psi(-p^\gamma) - \psi(-(p+1)^\gamma)) \ll \frac{x^\gamma}{\log^2 x}.$$

To bound $V_1(x)$ we split the sum over $n \leq x$ into $O(\log x)$ dyadic intervals of the form $(N, 2N]$ with $N \ll x$ and apply Lemma 6, obtaining that

$$\begin{aligned} V_1(x) &\ll \log x \cdot \max_{N \leq x} \left(\frac{1}{\log N} \max_{N_1 \leq 2N} \left| \sum_{N < n \leq N_1} \Lambda(n) X_{\mathcal{I}}(\{an + b\}) n^{\gamma-1} \right| + N^{1/2} \right) \\ &\ll x^{\gamma-1} \log x \cdot \max_{N \leq x} \max_{N_1 \leq 2N} \left| \sum_{N < n \leq N_1} X_{\mathcal{I}}(\{an + b\}) \right| + x^{1/2} \log x. \end{aligned}$$

Since $|\mathcal{I}| = 4\Delta$, it follows from the definition (2) and Lemma 1 that

$$V_1(x) \ll \Delta x^\gamma \log x + x^{\gamma - \frac{1}{\tau} + o(1)} \quad (x \rightarrow \infty).$$

Therefore,

$$V(\mathcal{I}, x) \ll \Delta x^\gamma \log x + \frac{x^\gamma}{\log^2 x}. \quad (6)$$

Now let $K \geq \Delta^{-1}$ be a large real number, and let Ψ_K be the trigonometric polynomial defined by

$$\Psi_K(t) = \sum_{|k| \leq K} g(k) e(kt). \quad (7)$$

Using (3) it is clear that the estimate

$$\Psi(t) = \Psi_K(t) + O(K^{-1} \Delta^{-1}) \quad (8)$$

holds uniformly for all $t \in \mathbb{R}$. Combining (8) with (5) and taking into account (6) we derive that

$$\pi_{\alpha, \beta}^{(c)}(x) = \sum_{p \leq x} \Psi_K(ap + b) \mathcal{X}^{(c)}(p) + O(E(x)),$$

where

$$E(x) = \Delta x^\gamma \log x + \frac{x^\gamma}{\log^2 x} + K^{-1} \Delta^{-1} \sum_{p \leq x} \mathcal{X}^{(c)}(p).$$

For fixed $A \in (0, 1)$ we put

$$\Delta = x^{-A/2} \quad \text{and} \quad K = x^A.$$

Note that our previous application of Lemma 3 to deduce (5) is justified. Use these values of Δ and K along with (4) we obtain that

$$E(x) \ll x^{\gamma - A/2} \log x + \frac{x^\gamma}{\log^2 x} + \frac{x^{\gamma - A/2}}{\log x} \ll \frac{x^\gamma}{\log^2 x}.$$

Using the definition (7) it therefore follows that

$$\pi_{\alpha, \beta}^{(c)}(x) = \sum_{p \leq x} \sum_{|k| \leq x^A} g(k) \mathbf{e}(kap + kb) \mathcal{X}^{(c)}(p) + O\left(\frac{x^\gamma}{\log^2 x}\right). \quad (9)$$

Next, using Lemma 11 we express the double sum in (9) as $\sum_1 + \sum_{2,1} + \sum_{2,2}$ with

$$\begin{aligned}\sum_1 &= g(0) \sum_{p \leq x} \mathcal{X}^{(c)}(p), \\ \sum_{2,1} &= \sum_{\substack{k \neq 0 \\ |k| \leq x^A}} g(k) \sum_{p \leq x} \mathbf{e}(kap + kb) (\gamma p^{\gamma-1} + O(p^{\gamma-2})), \\ \sum_{2,2} &= \sum_{\substack{k \neq 0 \\ |k| \leq x^A}} g(k) \sum_{p \leq x} \mathbf{e}(kap + kb) \{ \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \}.\end{aligned}$$

Recalling that $g(0) = \alpha^{-1}$ we have

$$\sum_1 = \alpha^{-1} \sum_{p \leq x} \mathcal{X}^{(c)}(p) = \frac{x^\gamma}{\alpha c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right),$$

which provides the main term in our estimation of $\pi_{\alpha,\beta}^{(c)}(x)$.

To bound $\sum_{2,1}$ we follow the method used above to bound $V(\mathcal{I}, x)$ and use partial summation together with (3) to conclude that

$$\sum_{2,1} \ll x^{\gamma-1} \log x \sum_{\substack{k \neq 0 \\ |k| \leq x^A}} \frac{1}{|k|} \max_{N \leq x} \left(\frac{1}{\log N} \max_{N' \leq 2N} \left| \sum_{N \leq n \leq N'} \Lambda(n) \mathbf{e}(k\alpha^{-1}n) \right| + 1 \right)$$

Assuming as we may that $0 < A < 1/(8\tau)$, by Lemma 7 it follows that there exists $\eta \in (0, 1)$ such that the bound

$$\max_{N \leq x} \left(\frac{1}{\log N} \max_{N' \leq 2N} \left| \sum_{N \leq n \leq N'} \Lambda(n) \mathbf{e}(k\alpha^{-1}n) \right| \right) \ll x^{1-\eta}$$

holds uniformly for $|k| \leq x^A$, $k \neq 0$. Consequently, we derive the bound

$$\sum_{2,1} \ll (x^{\gamma-1} x^{1-\eta} + x^{\gamma-1}) \log^2 x \ll \frac{x^\gamma}{\log^2 x},$$

which is acceptable.

To complete the proof it suffices to show that $\sum_{2,2} \ll x^\gamma / \log^2 x$. To accomplish this task we use the method in [4, pp. 47–53]. Denote

$$\sum_3 = \sum_{p \leq x} \mathbf{e}(kap + kb) \{ \psi(-(p+1)^\gamma) - \psi(-p^\gamma) \}.$$

It is enough to show that the bound $\sum_3 \ll x^{\gamma-\varepsilon}$ holds with some $\varepsilon > 0$ uniformly for k , for then we have by (3):

$$\sum_{2,2} \ll \sum_{\substack{k \neq 0 \\ |k| \leq x^A}} \frac{1}{|k|} \cdot x^{\gamma-\varepsilon} \ll x^{\gamma-\varepsilon} \log x \ll \frac{x^\gamma}{\log^2 x}.$$

By Lemma 4, for any $H \geq 1$ we can write

$$\sum_3 = \sum_4 + O(\sum_5),$$

where

$$\begin{aligned} \sum_4 &= \sum_{p \leq x} \sum_{0 < |h| \leq H} a_h (\mathbf{e}(kap + kb + h(p+1)^\gamma) - \mathbf{e}(kap + kb + hp^\gamma)), \\ \sum_5 &= \sum_{n \leq x} \sum_{|h| \leq H} b_h (\mathbf{e}(kan + kb + h(n+1)^\gamma) + \mathbf{e}(kan + kb + hn^\gamma)), \end{aligned}$$

with some numbers a_h, b_h that satisfy $a_h \ll |h|^{-1}$ and $b_h \ll H^{-1}$. Thus, it suffices to show that the bounds $\sum_4 \ll x^{\gamma-\varepsilon}$ and $\sum_5 \ll x^{\gamma-\varepsilon}$ hold with an appropriate choice of H . To this end, we put

$$H = x^{1-\gamma+2\varepsilon}.$$

First, we consider \sum_5 . The contribution from $h = 0$ is

$$2 \sum_{n < x} b_0 \mathbf{e}(kan + kb) \ll b_0 |ka|^{-1} \ll 1. \quad (10)$$

Suppose that $N \leq x$ and $N_1 \sim N$. We denote

$$S_j = \sum_{N < n \leq N_1} \sum_{0 < |h| \leq H} b_h \mathbf{e}(kan + kb + h(n+j)^\gamma).$$

To bound the part that $h \neq 0$, it suffices to show that $S_j \ll x^{1-\varepsilon}$ for $j = 0$ or 1 . By a shift of n , we have

$$S_j \ll \sum_{N < n \leq N_1} H^{-1} \sum_{0 < h \leq H} \mathbf{e}(kan + hn^\gamma).$$

Using Lemma 8 with the choice of $\lambda = hN^{\gamma-2}$, we obtain

$$\begin{aligned} S_j &\ll H^{-1} \sum_{0 < h \leq H} (N(hN^{\gamma-2})^{1/2} + (hN^{\gamma-2})^{-1/2}) \\ &\ll (x^{1-\gamma+2\varepsilon})^{1/2} x^{\gamma/2} + (x^{1-\gamma+2\varepsilon})^{-1/2} x^{1-\gamma/2} \ll x^{1/2+2\varepsilon}. \end{aligned}$$

Then summing over N , adding the part that $h = 0$ from (10) and recalling that $\gamma > 1/2$, we see that the bound

$$\sum_5 \ll x^{1/2+2\varepsilon} \log x + 1 \ll x^{\gamma-\varepsilon}$$

holds if the parameter ε is sufficiently small, which we can assume.

To bound \sum_4 we apply Lemma 6 and split the sum into $O(\log x)$ dyadic intervals of $(N, N_1]$ to derive the bound

$$\begin{aligned} &\sum_{N < p \leq N_1} \sum_{0 < |h| \leq H} a_h (\mathbf{e}(kap + kb + h(p+1)^\gamma) - \mathbf{e}(kap + kb + hp^\gamma)) \\ &\ll \frac{N^{\gamma-1}}{\log N} \max_{N_2 \leq 2N} \left| \sum_{1 \leq h \leq H} \sum_{N < n \leq N_2} \Lambda(n) \mathbf{e}(kan + kb + hn^\gamma) \right| + N^{1/2}. \end{aligned}$$

Summing over N and taking into account that $\gamma > 1/2$, we obtain the desired bound $\sum_4 \ll x^\gamma / \log^2 x$ (hence also $\sum_3 \ll x^\gamma / \log^2 x$) provided that

$$\sum_{1 \leq h \leq H} \sum_{N < n \leq N_2} \Lambda(n) \mathbf{e}(kan + kb + hn^\gamma) \ll x^{1-\varepsilon}. \quad (11)$$

Using Lemma 5, we can express the sum on the left side of (11) as

$$\sum_{1 \leq h \leq H} (-S_{1,h} + S_{2,h} - S_{3,h}),$$

where

$$\begin{aligned}
S_{1,h} &= \sum_{m \leq UV} \sum_{N/m \leq n \leq N_2/m} \tilde{a}(m) \mathbf{e}(kamn + kb + hm^\gamma n^\gamma), \\
S_{2,h} &= \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} \mu(m) (\log n) \mathbf{e}(kamn + kb + hm^\gamma n^\gamma), \\
S_{3,h} &= \sum_{V < n < N_2/U} \sum_{\substack{N/n \leq m \leq N_2/n \\ m > U}} \tilde{b}(n) \Lambda(m) \mathbf{e}(kamn + kb + hm^\gamma n^\gamma),
\end{aligned}$$

and the functions \tilde{a} and \tilde{b} are given by

$$\tilde{a}(m) = \sum_{\substack{cd=m \\ c \leq U \\ d \leq V}} \Lambda(c) \mu(d) \quad \text{and} \quad \tilde{b}(n) = \sum_{\substack{d|n \\ d \leq V}} \mu(d).$$

To establish (11) it suffices to show that

$$\sum_{1 \leq h \leq H} S_{j,h} \ll x^{1-\varepsilon} \quad (j = 1, 2, 3). \quad (12)$$

We turn to the problem of bounding $S_{1,h}$, $S_{2,h}$ and $S_{3,h}$. The sum $S_{2,h}$ is of type I, and $S_{3,h}$ is of type II. To bound $S_{1,h}$ we write it in the form $S_{4,h} + S_{5,h}$, where $S_{4,h}$ is a type I sum and $S_{5,h}$ is a type II sum. To simplify the calculation, we take

$$V = N^{3/7} \quad \text{and} \quad U = N^{1/7}.$$

Since $V \ll N^{3/7}$, we apply Lemma 9 to bound the sum $S_{2,h}$.

$$\begin{aligned}
\sum_{1 \leq h \leq H} S_{2,h} &\ll \sum_{1 \leq h \leq H} \log N \left| \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} \mathbf{e}(kamn + hm^\gamma n^\gamma) \right| \\
&\ll \sum_{1 \leq h \leq H} \log N (h^{1/2} N^{3/7+\gamma/2} + h^{-1/2} N^{1-\gamma/2}) \\
&\ll x^{27/14-\gamma+3\varepsilon} + x^{3/2-\gamma+\varepsilon} \ll x^{1-\varepsilon}
\end{aligned}$$

if assuming $\gamma > \frac{13}{14}$.

The sum $S_{3,h}$ can be split into $\ll \log^2 N$ subsums of the form

$$\sum_{X \leq m \leq 2X} \sum_{\substack{Y \leq n \leq 2Y \\ N \leq mn \leq N_1}} \alpha(m) \beta(n) \mathbf{e}(k\alpha^{-1}mn + hm^\gamma n^\gamma).$$

It suffices to consider the special case that $V < Y \leq N^{1/2}$ and $N^{1/2} < X \leq N/V$. Applying Lemma 10 (taking into account the estimates $\alpha(m) \ll N^{\varepsilon/2}$ and $\beta(n) \ll N^{\varepsilon/2}$) each subsum is

$$\ll (h^{-1/4} N^{1-\gamma/4} + h^{1/6} N^{16/21+\gamma/6} + N^{11/14}) N^\varepsilon.$$

Therefore, the bound

$$\begin{aligned} \sum_{1 \leq h \leq H} S_{3,h} &\ll (H^{3/4} N^{1-\gamma/4} + H^{7/6} N^{16/21+\gamma/6} + H N^{11/14}) N^\varepsilon \\ &\ll ((x^{1-\gamma+2\varepsilon})^{3/4} x^{1-\gamma/4} + (x^{1-\gamma+2\varepsilon})^{7/6} x^{16/21+\gamma/6} + (x^{1-\gamma+2\varepsilon}) x^{11/14}) x^\varepsilon \\ &\ll (x^{7/4-\gamma} + x^{27/14-\gamma} + x^{25/14-\gamma}) x^{4\varepsilon} \ll x^{1-\varepsilon} \end{aligned}$$

under our hypothesis that $\gamma > \frac{13}{14}$.

Finally, to derive the required bound $S_{1,h} \ll x^{1-\varepsilon}$ we write

$$S_{1,h} = S_{4,h} + S_{5,h},$$

where

$$\begin{aligned} S_{4,h} &= \sum_{m \leq V} \sum_{N/m \leq n \leq N_2/m} a(m) \mathbf{e}(kamn + kb + hm^\gamma n^\gamma), \\ S_{5,h} &= \sum_{V < m \leq UV} \sum_{N/m \leq n \leq N_2/m} a(m) \mathbf{e}(kamn + kb + hm^\gamma n^\gamma). \end{aligned}$$

Since $a(m) \leq \log m$ the methods used above to bound $S_{2,h}$ and $S_{3,h}$ can be applied to $S_{4,h}$ and $S_{5,h}$, respectively, to see that the bounds

$$\sum_{1 \leq h \leq H} S_{j,h} \ll x^{1-\varepsilon} \quad (j = 4, 5). \quad (13)$$

hold under our hypothesis that $\gamma > \frac{13}{14}$. This establishes (13), and the theorem is proved.

4 Remarks

We note that both [1, Theorem 7] and [1, Theorem 8] can be improved using Lemma 9 and Lemma 10 instead of [1, Lemma 24] and [1, Lemma 25], respectively. The range of c in [1, Theorem 7] can be extended from $(1, \frac{147}{145})$ to $(1, \frac{571}{561})$, with a small improvement of 0.004. For [1, Theorem 8], the range of c is improved from $(1, \frac{18}{17})$ to $(1, \frac{14}{13})$ and the error term is improved from $O(x^{17/39+7\gamma/13+\varepsilon})$ to $O(x^{3/7+7\gamma/13+\varepsilon})$.

It would be interesting to see whether the range of c in the statement of Theorem 1 can be improved using more sophisticated methods to improve our type II estimates. With more work, it should be possible to remove our assumption that α is of finite type. For the sake of simplicity, these ideas have not been pursued in the present paper.

Acknowledgement. The author would like to thank his advisor, William Banks, for suggesting this work and for several helpful discussions.

References

- [1] R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski and A. J. Wein-gartner, ‘Piatetski-Shapiro sequences,’ *Acta Arith.* **157** (2013), no. 1, 37–68.
- [2] W. D. Banks and I. E. Shparlinski, ‘Prime numbers with Beatty se-quences,’ *Colloq. Math.* **115** (2009), no. 2, 147–157.
- [3] H. Davenport *Multiplicative number theory*. Graduate Texts in Mathe-matics, **74**. Springer-Verlag, New York-Berlin, 1980.
- [4] S. W. Graham and G. Kolesnik, *Van der Corput’s method of exponential sums*. London Mathematical Society Lecture Note Series, **126**. Cam-bridge University Press, Cambridge, 1991.

- [5] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*. Pure and Applied Mathematics. Wiley-Interscience, New York-London-Sydney, 1974.
- [6] I. I. Piatetski-Shapiro, ‘On the distribution of prime numbers in the sequence of the form $\lfloor f(n) \rfloor$,’ *Mat. Sb.* **33** (1953), 559–566.
- [7] P. Ribenboim *The new book of prime number records*. Springer-Verlag, New York, 1996.
- [8] J. Rivat and J. Wu, ‘Prime numbers of the form $\lfloor n^c \rfloor$,’ *Glasg. Math. J.* **43** (2001), no. 2, 237–254.
- [9] J. D. Vaaler, ‘Some extremal problems in Fourier analysis,’ *Bull. Amer. Math. Soc.* **12** (1985), 183–216.
- [10] I. Vinogradov, *The method of trigonometrical sums in the theory of numbers*. Dover Publications, Inc., Mineola, NY, 2004.
- [11] D. Leitmann and D. Wolke, ‘Primzahlen der Gestalt $\lfloor n^\Gamma \rfloor$ in arithmetischen progressionen’, (German) *Arch. Math.* (Basel) **25** (1974), 492–494.